Epipolar Geometry

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Reference: Multiple View Geometry in Computer Vision, by *Hartley and Zisserman*
Some slides modified from Marc Pollefeys at UNC
http://www.cs.unc.edu/Research/vision/comp256/

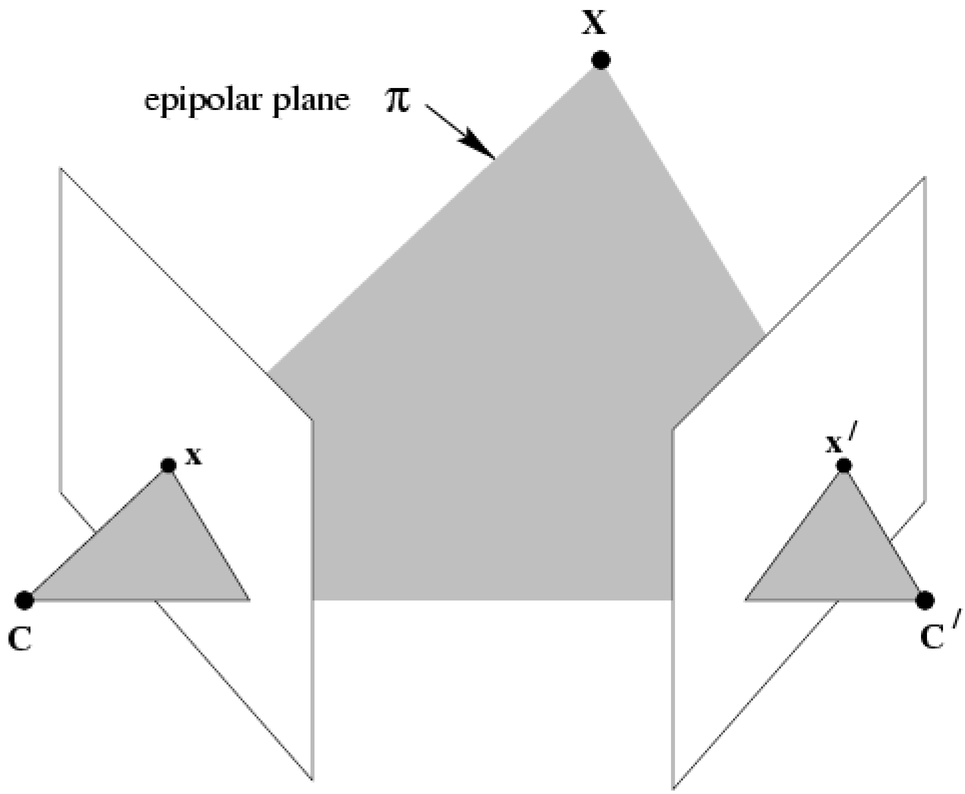
Three questions:

(i) **Correspondence geometry:** Given an image point x in the first view, how does this constrain the position of the corresponding point x' in the second image?

(ii) **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, $i=1,\dots,n$, what are the cameras P and P' for the two views?

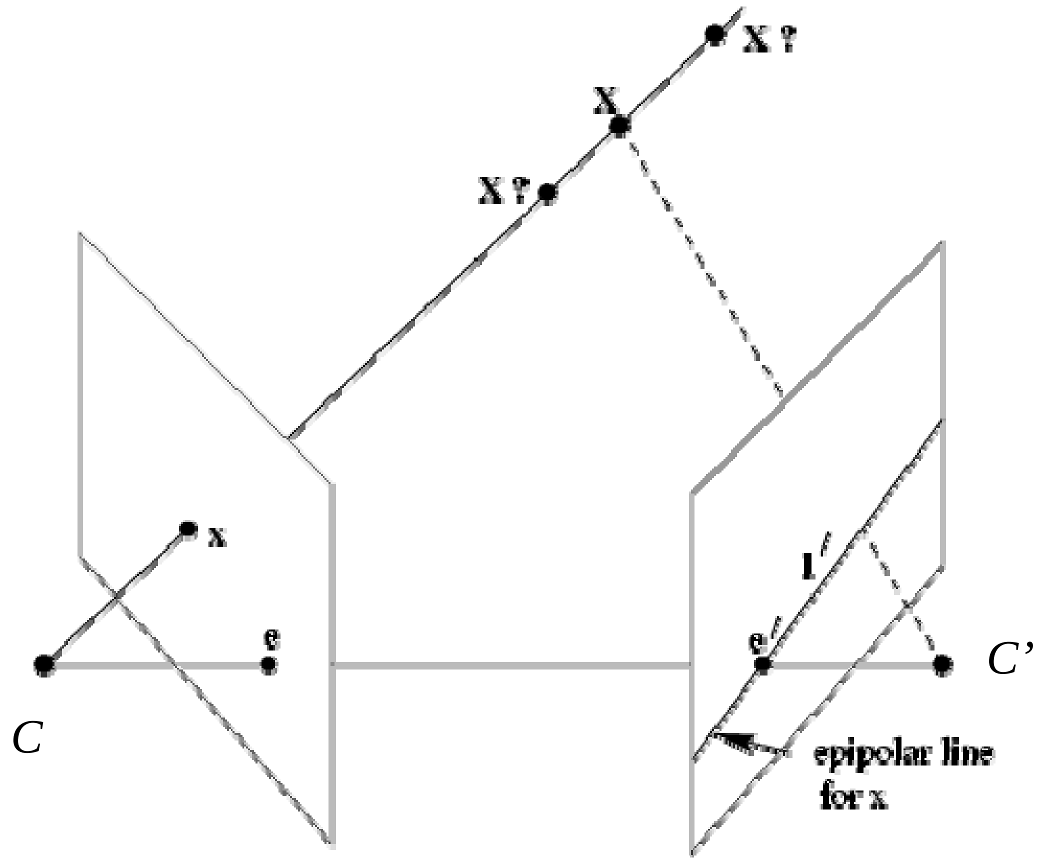
(iii) **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

The epipolar geometry



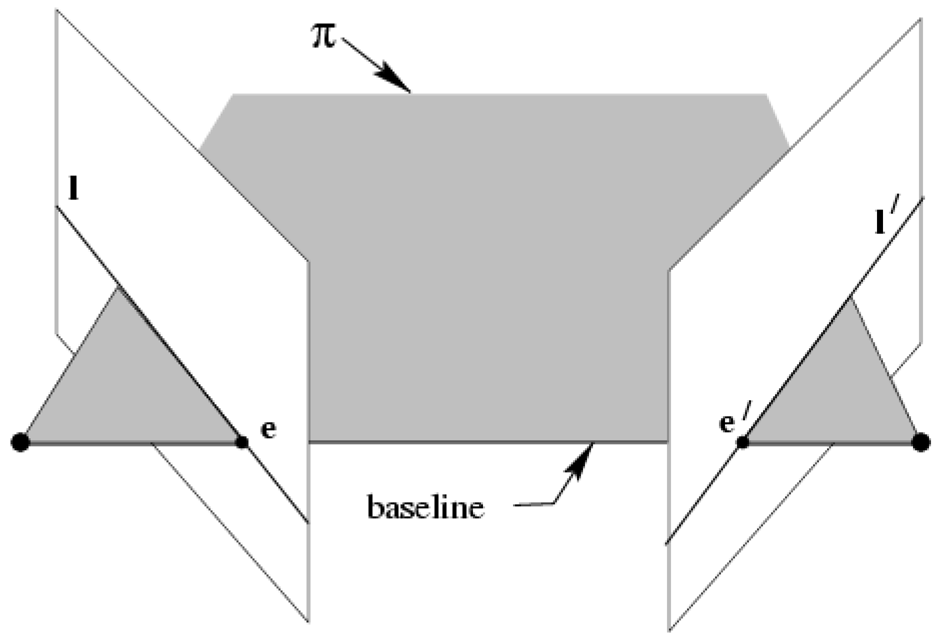
C,C',x,x' and X are coplanar

The epipolar geometry



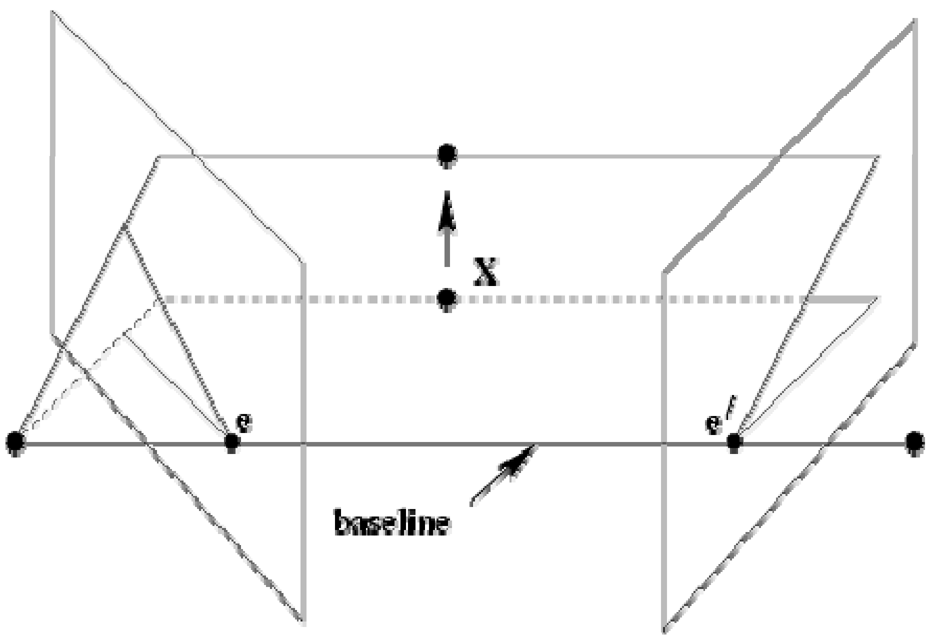
What if only C,C’,x are known?

The epipolar geometry



All points on π project on l and l'

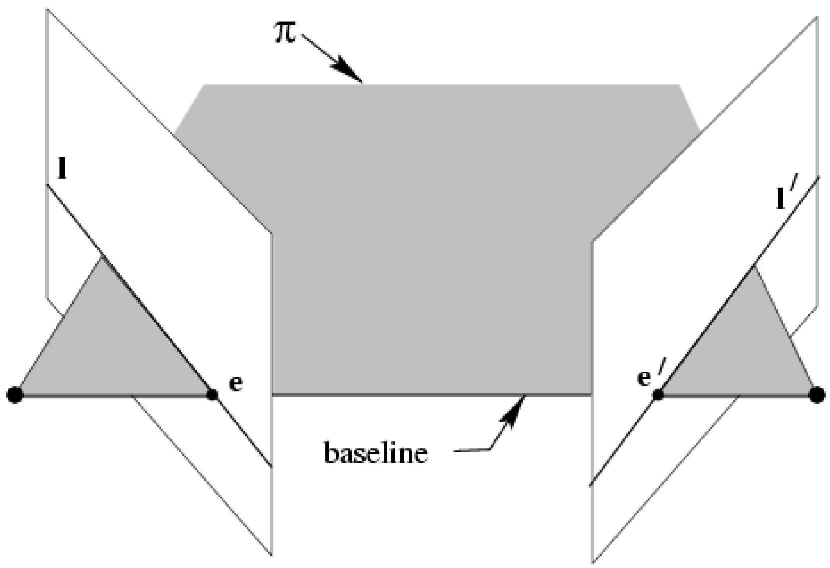
The epipolar geometry



Family of planes π through the camera centers
and families of lines l and l' Intersection in e and e'

The epipolar geometry

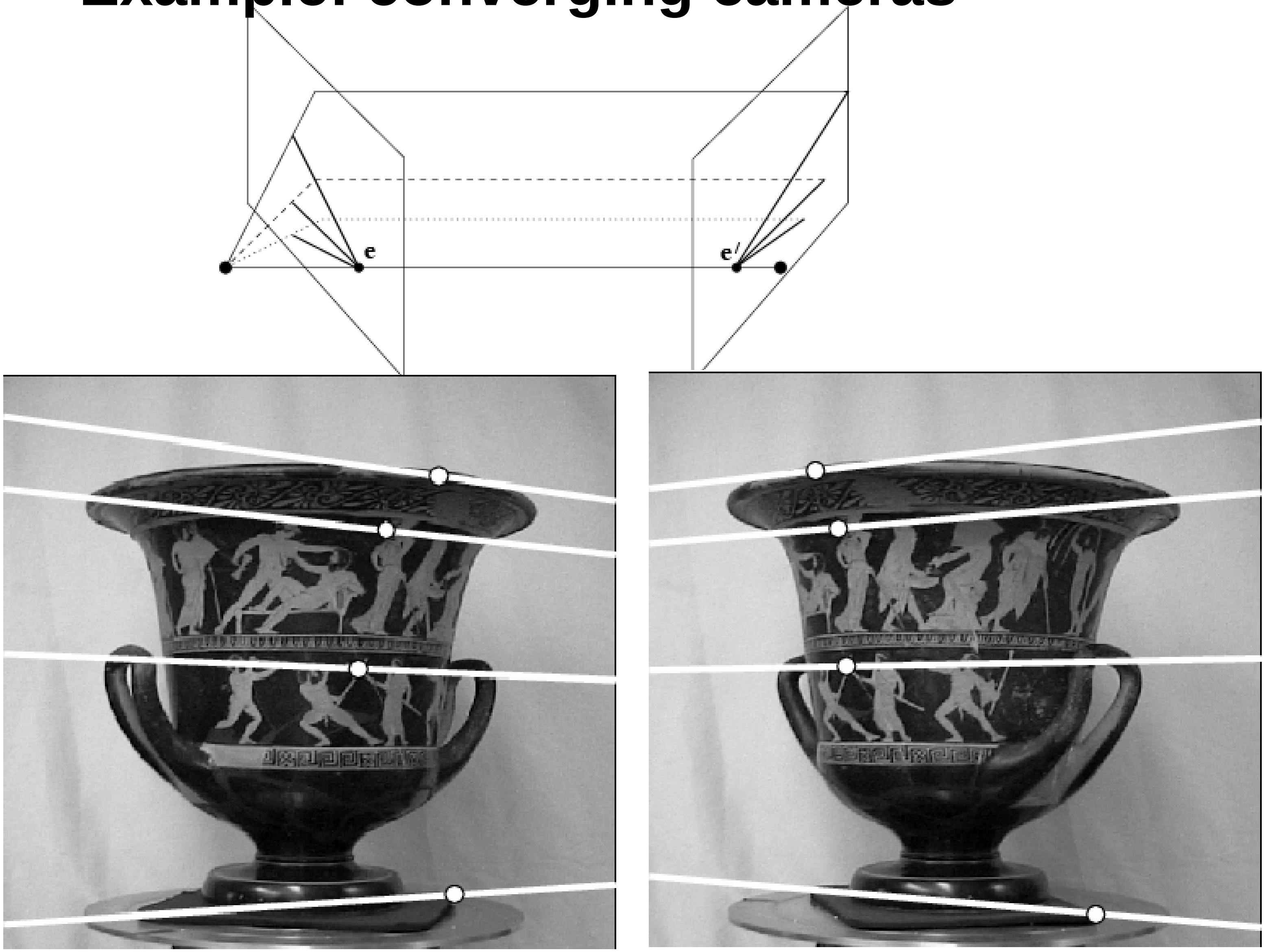
Three interpretations of epipoles e and e'
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction



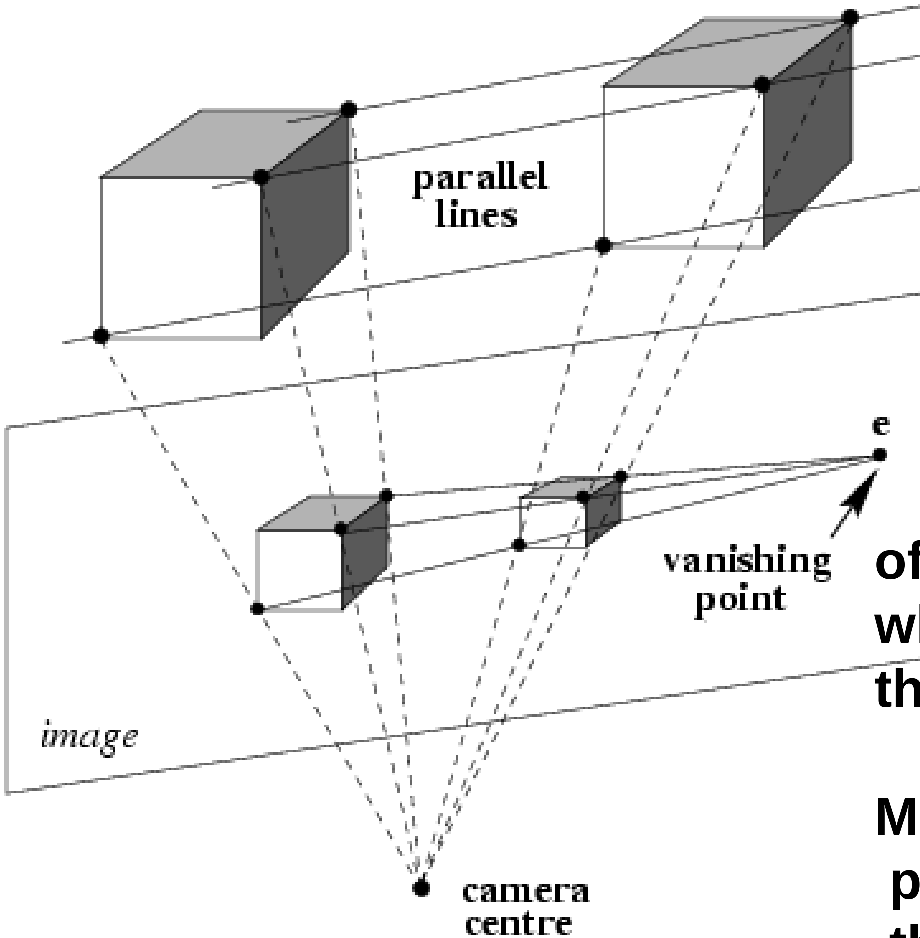
Why and what is the 1 DOF?

an epipolar plane = plane containing baseline (1-D family)
an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)
All epipolar lines intersect at the epipole.

Example: converging cameras



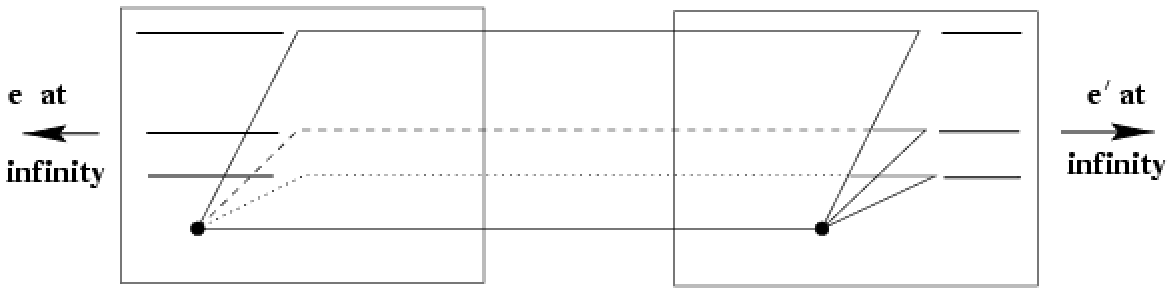
Example: pure translation

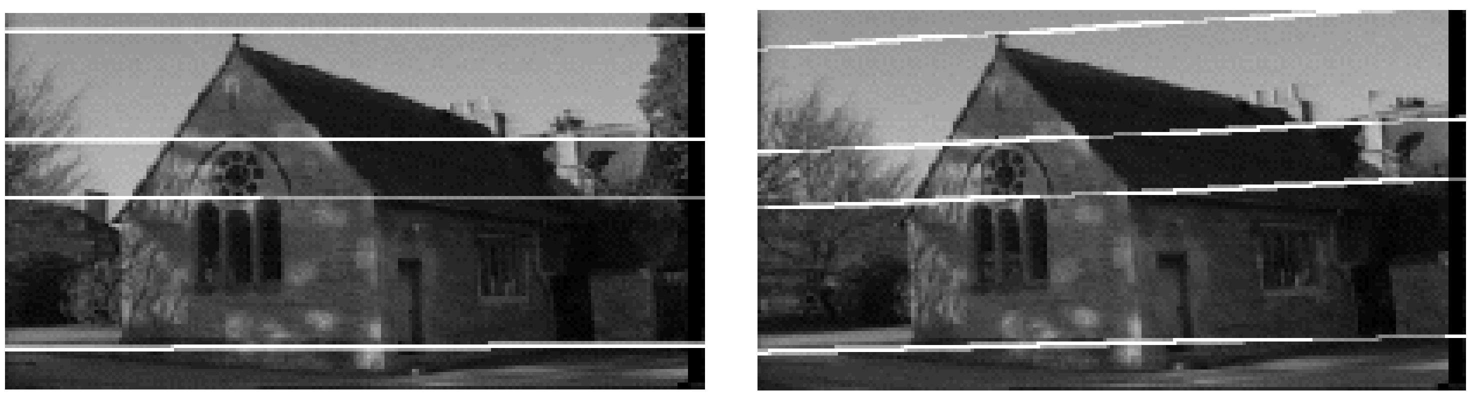


of the translation direction, which is also the directions of the parallel lines.

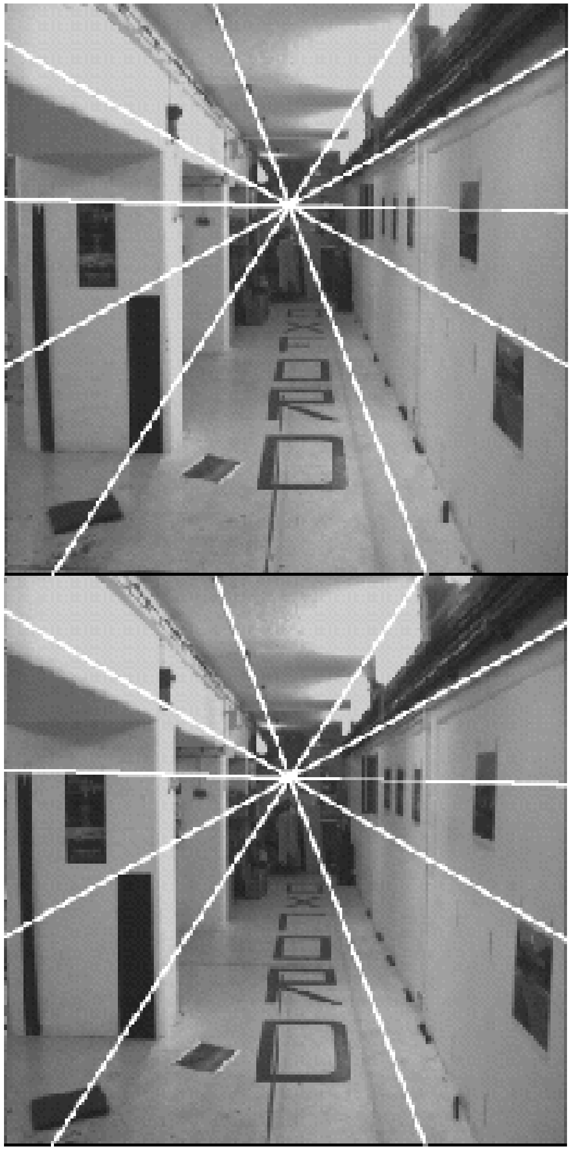
Motion vectors resulting from pure translation converges to the epipole.

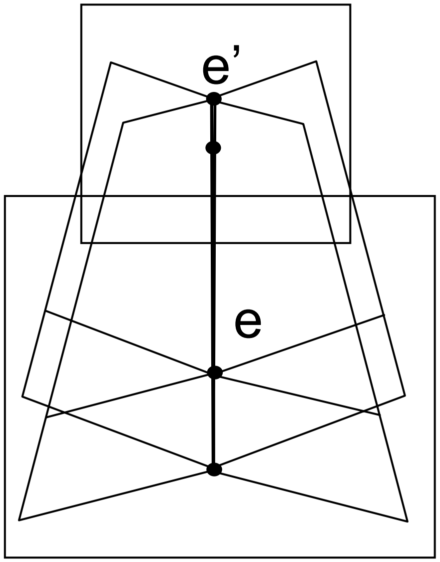
Example: motion parallel with image plane



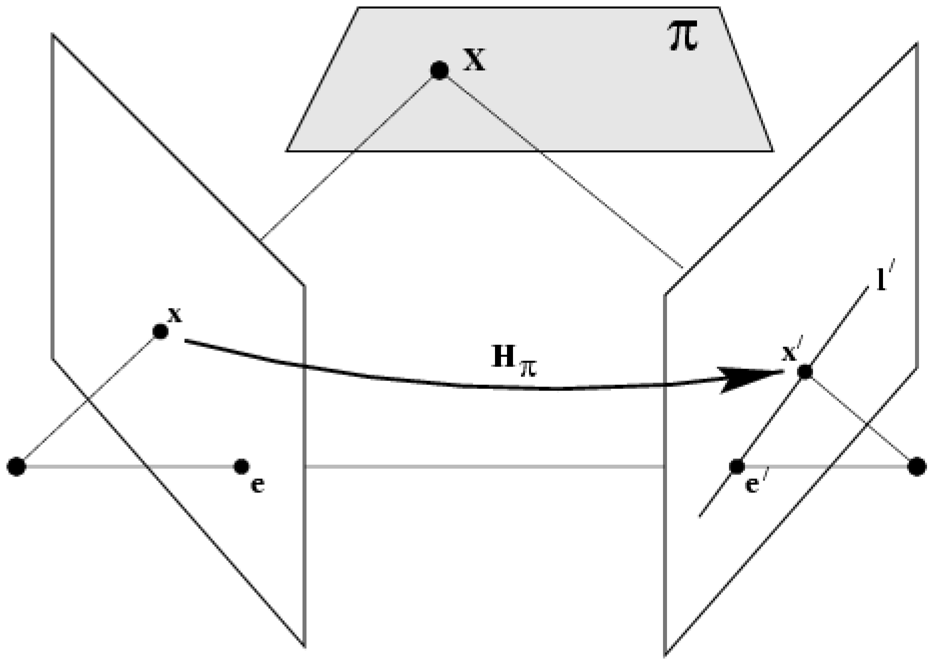


Example: forward motion





Planar scene and homography



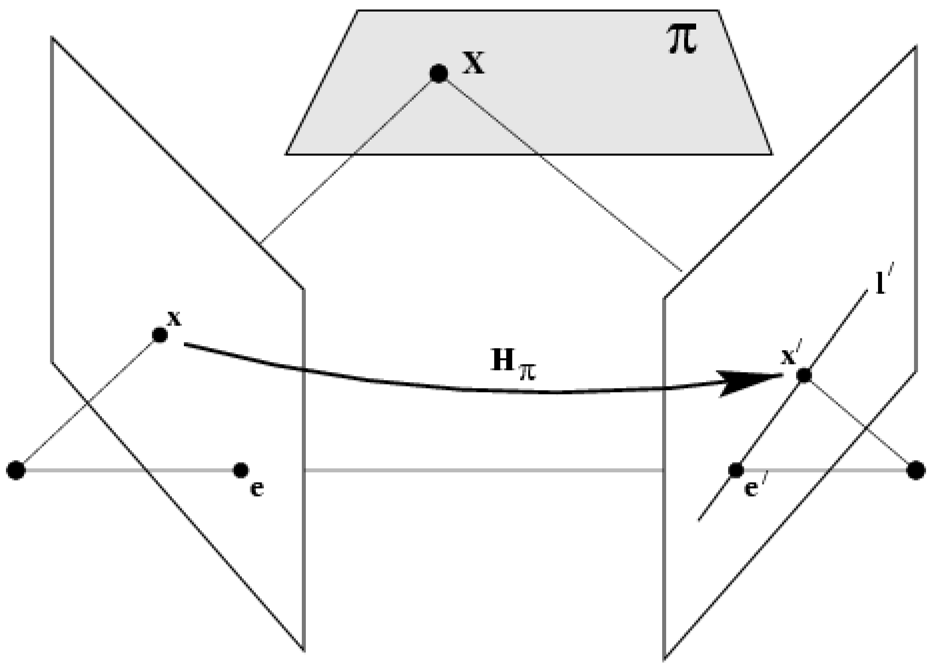
$$P=[I,0],\ P'=[A,a],\ \pi^T=(v^T,1) \Rightarrow H_\pi = A - av^T \text{ s.t. } x'= H_\pi x$$

$$P=K[I,0],\ P'=K'[R,t],\ \pi^T=(n^T,d) \Rightarrow H_\pi = K'(R-tn^T/d)K^{-1} \text{ s.t. } x'= H_\pi x$$

The fundamental matrix F

geometric interpretation

When π is π_∞, $H_\infty = K'RK^{-1}$



Skew symmetric matrix, $A^T = -A$
rank 2, e is the right null vector.

$$[e]_\times x = \begin{bmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{bmatrix} x$$

$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

mapping from 2-D to 1-D family (rank 2)

F given projective cameras

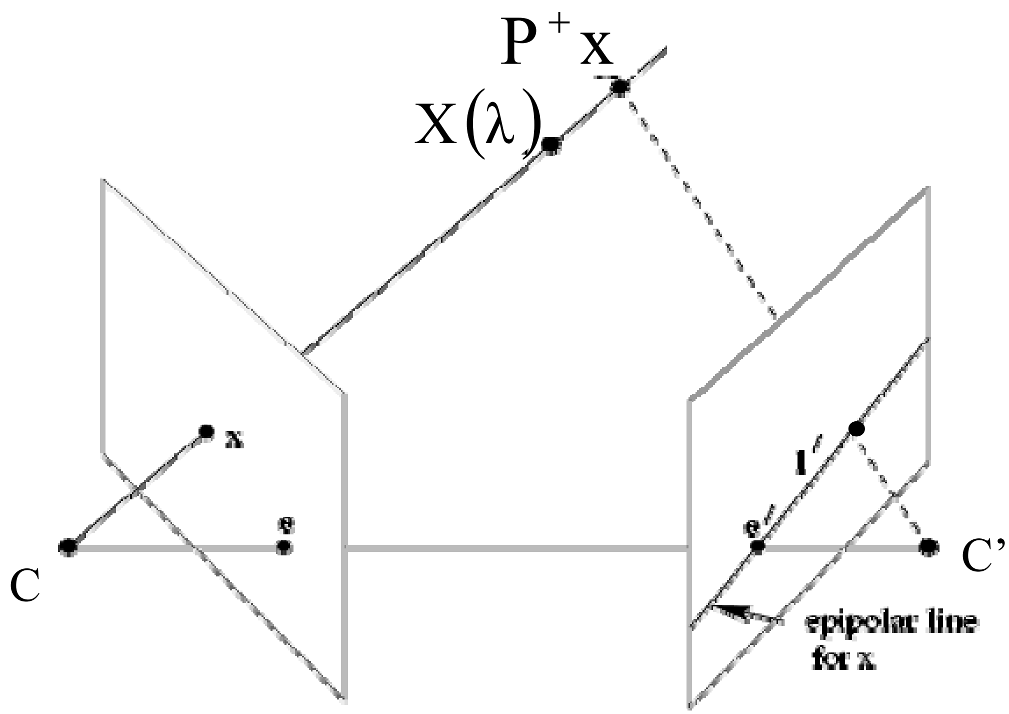
- The fundamental matrix corresponding to a pair of camera matrices P=[I|0], P'=[M,m] is

$$F = [m]_x M$$

The fundamental matrix F

Algebraic interpretation $X(\lambda) = P^{+}x + \lambda C$ $\quad (PP^{+} = I)$

$$l' = P'C \times P'P^{+}x \qquad F = [e']_{\times} P'P^{+}$$



(note: doesn't work for C=C' $\Rightarrow$ F=0) why?

The fundamental matrix F

When the space is Euclidean, P=K[I|0], P'=K'[R|t]

- **First camera system used as the world frame.**
- **t, first camera center in the second camera frame.**
- **x' is the projection of the direction point of CX on π_∞**

For any invertible H,

$$[Hx]_\times = H^{-T}[x]_\times H^{-1}$$

$$[Hx]_\times(Hy) = H^{-T}[x]_\times y$$

$$Fx = l' = [e']_\times x' = [K't]_\times K'Rd$$

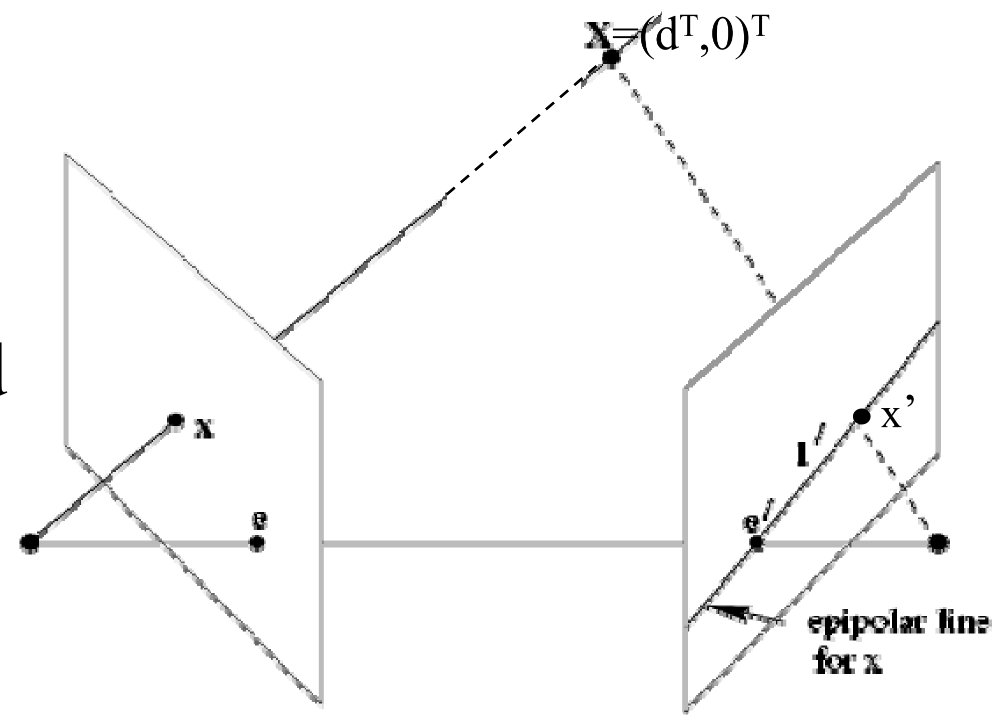
$$= [K't]_\times K'RK^{-1}x$$

$$= K'^{-T}[t]_\times RK^{-1}x$$

$$F^T x' = l = [e]_\times x = [KR^T t]_\times Kd$$

$$= [KR^T t]_\times K(K'R)^{-1}x'$$

$$= K^{-T}R^T[t]_\times K'^{-1}x'$$



The fundamental matrix F

correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

$$x'^{T} F x = 0 \qquad \left(x'^{T} l' = 0\right)$$

The fundamental matrix F

F is the unique 3x3 rank 2 matrix that satisfies $x'^T F x = 0$ for all $x \leftrightarrow x'$

(i) Transpose: if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)

(ii) Epipolar lines: $l' = Fx$ & $l = F^T x'$

(iii) Epipoles: on all epipolar lines, thus $e'^T F x = 0, \forall x \Rightarrow e'^T F = 0$, similarly $Fe = 0$, right null space of F

(iv) F has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)

Projective transformation and invariance

Derivation based purely on projective concepts, such as the intersection of lines and planes, and in the algebraic development only the linear mapping of the projective camera between world and image points.

$$\hat{x} = Hx, \hat{x}' = H'x' \Rightarrow \hat{F} = H'^{-T} F H^{-1}$$

F invariant to transformations of projective 3-space since the matching images points are unchanged.

$$x = PX = (PH)(H^{-1}X) = \hat{P}\hat{X} \qquad x' = P'X = (P'H)(H^{-1}X) = \hat{P}'\hat{X}$$

$$F = [e']_{\times} P' P^{+}$$

$$\text{If } (P, P') \mapsto F, \text{ then } (PH, P'H) \mapsto F, \text{ for any invertable H.}$$

$(P, P') \mapsto F$ unique $\qquad F \mapsto (P, P')$ not unique

canonical form

$$P = [I \mid 0]$$
$$P' = [M \mid m]$$

$$F = [m]_{\times} M$$

Projective ambiguity of cameras given F

previous slide: at least a projective ambiguity
this slide: this is the only ambiguity and not more!

We need to show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'), there exists a projective transformation H so that $\tilde{P}$=PH and $\tilde{P}$'=P'H.

$$P = [I \mid 0], P' = [A \mid a], \tilde{P} = [I \mid 0], \tilde{P}' = [\tilde{A} \mid \tilde{a}]$$

$$F = [a]_\times A = [\tilde{a}]_\times \tilde{A}$$

<u>lemma:</u> $\tilde{a} = ka \quad \tilde{A} = k^{-1}\left(A + av^T\right)$

$$aF = a[a]_\times A = 0 = \tilde{a}F \overset{\text{rank 2}}{\Longrightarrow} \tilde{a} = ka$$

$$[a]_\times A = [\tilde{a}]_\times \tilde{A} \Rightarrow [a]_\times \left(k\tilde{A} - A\right) = 0 \Rightarrow \left(k\tilde{A} - A\right) = av^T$$

$$H = \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}v^T & k \end{bmatrix} \quad P'H = [A \mid a]\begin{bmatrix} k^{-1}I & 0 \\ k^{-1}v^T & k \end{bmatrix} = [k^{-1}\left(A - av^T\right) \mid ka] = \tilde{P}'$$

(22-15=7, ok)

Canonical cameras given F

F matrix corresponds to P,P' iff $P'^T F P$ is skew-symmetric.

$$\left(X^T P'^T F P X = 0, \forall X\right)$$

F matrix, S skew-symmetric matrix so that P' is full rank.

$$P = [I \mid 0] \quad P' = [SF \mid e']$$ (fundamental matrix=F)

$$\left([SF \mid e']^T F [I \mid 0] = \begin{bmatrix} F^T S^T F & 0 \\ e'^T F & 0 \end{bmatrix} = \begin{bmatrix} F^T S^T F & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Possible choice:

$$P = [I \mid 0] \quad P' = [[e']_\times F \mid e']$$

Canonical representation:

$$P = [I \mid 0] \quad P' = [[e']_\times F + e' v^T \mid \lambda e']$$

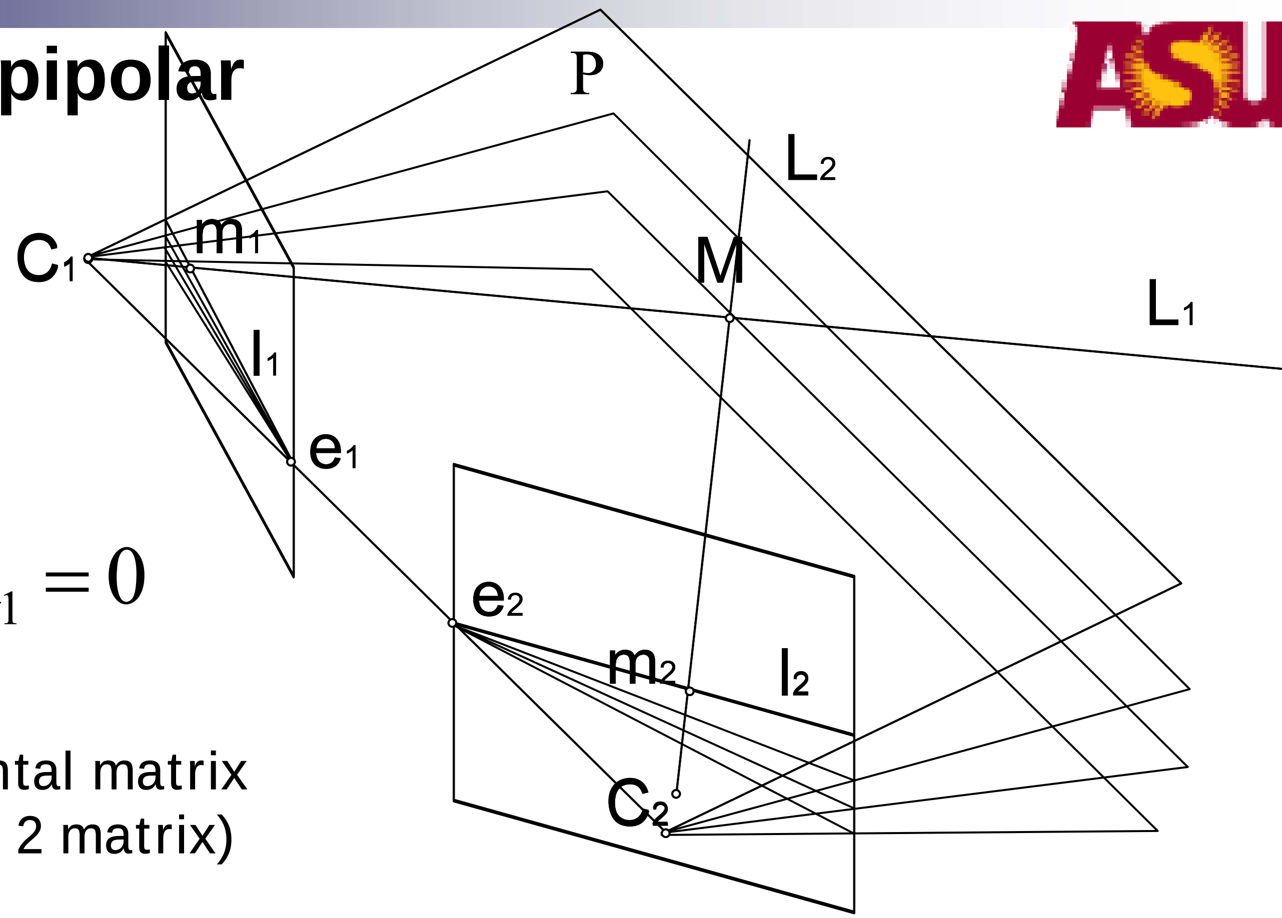
Application of Epipolar geometry

Underlying structure in set of matches for rigid scenes

$$\overbrace{m_2^T \mathbf{F}}^{\mathbf{l^T_1}}\ \overbrace{\mathbf{F}\, m_1}^{\mathbf{l_2}} = 0$$

$$m_2^T \mathbf{F}\, m_1 = 0$$

Fundamental matrix (3x3 rank 2 matrix)



1. Computable from corresponding points
2. Simplifies matching
3. Allows to detect wrong matches
4. Related to calibration

Canonical representation:

$$P = [I \,|\, 0] \quad P' = [[e']_\times F + e' v^T \,|\, \lambda e']$$

Estimation of the Fundamental Matrix

- Basic equation $x'^{T} F x = 0$
- The 8-point algorithm

$$\begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} f = 0$$

- Normalization
  - Translate origin to centroid
  - Scale to a $\sqrt{2}$ average distance to the origin
  - Applied independently on both images

$Af=0 \rightarrow$ f is the right null vector of A $\rightarrow \hat{F}$

$$\tilde{x} = Tx, \quad \tilde{x}' = T'x'$$

Centroid position

$$T = \begin{bmatrix} s & & -s\bar{x}_1 \\ & s & -s\bar{x}_2 \\ & & 1 \end{bmatrix}$$

$$s = \sqrt{2} / d$$

Average distance to the centroid

References: Hartley, R. I., In Defense of the Eight-Point Algorithm, *IEEE Trans. PAMI*, vol. 19, pp. 580-593, 1997

Zhang, Z.Y., Determining The Epipolar Geometry And Its Uncertainty: A Review, *IJCV(27)*, No. 2, March 1998, pp. 161-195.

Computation of the Fundamental Matrix

- Enforcing the Singularity (rank=2) Constraint

$$\hat{F} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$$\tilde{F} = [u_1 \quad u_2]\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

- De-normalization

$$\tilde{\mathrm{x}}'^{\mathrm{T}}\,\tilde{\mathrm{F}}\tilde{\mathrm{x}} = 0 \Leftrightarrow \mathrm{x}'^{\mathrm{T}}\,\mathrm{T}'^{\mathrm{T}}\,\tilde{\mathrm{F}}\mathrm{T}\mathrm{x} = 0$$

Hence, $\mathrm{F} = \mathrm{T}'^{\mathrm{T}}\,\tilde{\mathrm{F}}\mathrm{T}$

For any invertible H,

$$[\mathrm{Hx}]_\times = \mathrm{H}^{-\mathrm{T}}[\mathrm{x}]_\times \mathrm{H}^{-1}$$ **or i.e.** $$[\mathrm{Hx}]_\times(\mathrm{Hy}) = \mathrm{H}^{-\mathrm{T}}[\mathrm{x}]_\times \mathrm{y}$$

RANSAC- **Random Sample Consensus**

Objective

Robust fit of model to data set ***S*** which might contain outliers.

Algorithm

(i) Randomly select a sample of *s* data points from ***S*** and instantiate the model from this subset.

(ii) Determine the set of data points S_I which are within a distance threshold d_{th} of the model. The set S_I is the consensus set of samples and defines the inliers of ***S***.

(iii) If the subset of S_I is greater than some threshold *T*, re-estimate the model using all the points in S_I and terminate the process.

(iv) If the size of S_I is less than *T*, select a new subset and repeat the above.

(v) After *N* trials the largest consensus set S_I is selected, and the model is re-estimated using all the points in the subset S_I

Reference: Fischler, M. A. and Bolles, R. C., **Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography.** Comm. of the ACM, Vol 24, pp 381-395, 1981.

Determine N, the number of samples in RANSAC

Let p, be the prob. that at least one of random samples of s points is free from outliers.
Let w, be the prob. that a select data point is an inlier.

The prob. that all N sample sets contains outliers is

$$1 - P = (1 - w^s)^N$$

Prob that a sample set has no outliers.

$$P = 1 - (1 - w^s)^N \qquad N = \log(1-P)/\log(1-w^s)$$

Select N so that P can be greater than a prechosen value, e.g. 0.95, as follows.

$1 - (1 - w^s)^N > 0.95 \Rightarrow$

$(1 - w^s)^N < 0.05 \Rightarrow$

$N \log(1 - w^s) < \log 0.05 \Rightarrow$

$N > \log 0.05 / \log(1 - w^s)$